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Uniqueness of moving boundary for a heat conduction problem with nonlinear interface conditions

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ABSTRACT

In this paper, based on the maximum principle and the unique continuation theorem, we present a uniqueness result for a moving boundary of a heat problem in a multilayer medium with nonlinear interface conditions.

1. Introduction

The boundary identification problem for the Laplace equation or a heat equation arises in the ironmaking blast furnace where it is desired to monitor the corroded thickness of the accreted refractory wall based on the measurement of temperature and heat flux on an accessible part of boundary or some internal positions. This kind of problem is ill-posed in Hadamard's sense. That is, any small change on the input data can result in a dramatic change to the solution. Hence, a special regularization technique is necessary for stabilizing the computation. A number of numerical methods for determining a portion of steady state boundary for a heat conducting solid have been proposed in the Refs. [1–3]. However, for estimating a time-varying boundary in the heat conduction problem, as we know, not many papers can be found [4–7] in which the initial temperature should be used. Most of the papers mentioned above used an iterative method to reconstruct an unknown boundary. In [5], Fredman employed a direct method, called the method of lines, to calculate a moving boundary in one-dimensional heat conduction problem. Liu and Guerrier in [6] applied a domain embedding method for estimating the moving boundary in an inverse Stefan problem where solving an optimization problem by an iterative process is required. Thus the initial temperature data should be given in advance. In [4], Badia and Moutazaim constructed an identification method based on minimizing a Tikhonov functional by an iterative algorithm. In this paper we focus on the uniqueness of a moving boundary for a complicated multi-layers heat problem and will study numerical methods in the future work.

We note that Manselli and Vesella had proved the continuous dependence of moving boundary on noncharacteristic Cauchy data under an a priori information even without using the initial temperature [7]. Thus, for the boundary identification problem of heat equation, the initial condition is not necessary.

Moreover, in some practical problems, the considered body consists of several layers with different material properties such that the unknown temperature are discontinuous through the interfaces. For this motivation, we deal with a heat problem with a composite material in this paper. Because the temperature at the side of moving boundary is much more high than the temperature at another side, the interface condition between two different materials obeys the nonlinear Stefan–Boltzmann law. To our knowledge, such a problem has not been researched previously.

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In this paper, we give a research note on the uniqueness of moving boundary (if it exists) in a multilayer medium with nonlinear interface conditions. The related direct problem has been studied in the paper [8] in which Yang et al. proved that there is a unique classical solution for the direct problem.

2. The formulation of a boundary identification problem with nonlinear interface conditions

In this paper, we consider a heat conduction problem in a multilayer domain with a moving boundary $s(t)$. For simplicity, we use the following three layers problem as an example, which comes from a real world application.

The temperature distributions in each subdomain satisfy the following equations

$$\frac{\partial u_1}{\partial t}(x,t) = a_1^2 \frac{\partial^2 u_1}{\partial x^2}(x,t), \quad \text{in } D_1 = (0, l_1) \times (0, T), \tag{2.1}$$

$$\frac{\partial u_2}{\partial t}(x,t) = a_2^2 \frac{\partial^2 u_2}{\partial x^2}(x,t), \quad \text{in } D_2 = (l_1, l_2) \times (0, T), \tag{2.2}$$

$$\frac{\partial u_3}{\partial t}(x,t) = a_3^2 \frac{\partial^2 u_3}{\partial x^2}(x,t), \quad \text{in } D_3 = (l_2, s(t)) \times (0, T), \tag{2.3}$$

with Stefan–Boltzmann interface conditions

$$\lambda_1 \frac{\partial u_1}{\partial x}(l_1, t) = \sigma_1(u_2^4(l_1,t) - u_1^4(l_1,t)), \quad 0 \le t \le T, \tag{2.4}$$

$$\lambda_1 \frac{\partial u_1}{\partial x}(l_1, t) = \lambda_2 \frac{\partial u_2}{\partial x}(l_1, t), \quad 0 \le t \le T, \tag{2.5}$$

$$\lambda_2 \frac{\partial u_2}{\partial x}(l_2, t) = \sigma_2(u_3^4(l_2,t) - u_2^4(l_2,t)), \quad 0 \le t \le T, \tag{2.6}$$

$$\lambda_2 \frac{\partial u_2}{\partial x}(l_2, t) = \lambda_3 \frac{\partial u_3}{\partial x}(l_2, t), \quad 0 \le t \le T, \tag{2.7}$$

and boundary conditions at the fixed end $x = 0$

$$u_1(0,t) = u_0(t), \quad 0 \le t \le T, \tag{2.8}$$

$$\frac{\partial u_1}{\partial x}(0,t) = q_0(t), \quad 0 \le t \le T, \tag{2.9}$$

where $u_i(x,t),\ i = 1, 2, 3$ are the temperature distributions in each subdomain, T represents the maximum time of interest for the time evolution of the problem and heat coefficients $a_i, \lambda_i, \sigma_i, i = 1, 2, 3$ are positive constants.

The boundary identification problem of the heat problem is then to determine the boundary movement function $s(t)$ from a Dirichlet boundary condition

$$u_3(s(t), t) = u_M, \tag{2.10}$$

where $u_M > 0$ is a given constant indicating a fusion point of a medium.

In [7], the authors gave a conditional stability result for one phase case, from which, we know there is at most one moving boundary for the inverse boundary problem.

In this paper, for the multilayer case with the nonlinear interface conditions, we firstly prove a uniqueness result for the moving boundary.

3. The uniqueness of moving boundary for the boundary identification problem

Denote the parabolic boundary for each subdomain $D_i,\ i = 1, 2, 3$ as follows,

$$\begin{aligned} &\Gamma_1 = \{0 \le x \le l_1, t = 0\} \cup \{x = 0, x = l_1, 0 < t \le T\},\\ &\Gamma_2 = \{l_1 \le x \le l_2, t = 0\} \cup \{x = l_1, x = l_2, 0 < t \le T\},\\ &\Gamma_3 = \{l_2 \le x \le s(0), t = 0\} \cup \{x = l_2, x = s(t), 0 < t \le T\}, \end{aligned}$$

and function spaces

$$C_1^2(D_i) = \{u : D_i \to \mathcal{R} \mid u, u_{xx}, u_t \in C(D_i)\}, \quad i = 1, 2, 3,$$

and

$$C_0^1(\overline{D}_i) = \{u : \overline{D}_i \to \mathcal{R} \mid u, u_x \in C(\overline{D}_i)\}, \quad i = 1, 2, 3.$$

Then we have the following lemmas.

Lemma 3.1. *Let* $s(t) \in C[0,T], s(t) > l_2 > 0, t \in [0,T]$. *If* $u_i(x,t) \in C_1^2(D_i) \cap C(\overline{D}_i),\ i = 1,2,3$ *satisfy* (2.1)–(2.3), *then*

$$\max_{(x,t)\in\overline{D}_i} u_i(x,t) = \max_{(x,t)\in\Gamma_i} u_i(x,t), \quad i = 1,2,3, \tag{3.1}$$

and

$$\min_{(x,t)\in\overline{D}_i} u_i(x,t) = \min_{(x,t)\in\Gamma_i} u_i(x,t), \quad i = 1,2,3. \tag{3.2}$$

Proof. By the maximum principle in one domain, see, e.g., [9], it is easy to obtain the results in this lemma. □

In the following, we denote $I_1 = [0, l_1], I_2 = [l_1, l_2], I_3 = [l_2, s(0)]$.

Lemma 3.2 (*The Positivity of the Solution*). *Let* $s(t) \in C[0,T], s(t) > l_2 > 0, t \in [0,T]$. *Suppose* $u_i(x,t) \in C_1^2(D_i) \cap C_0^1(\overline{D}_i),\ i = 1,2,3$ *satisfy* (2.1)–(2.10). *If*

$$u_i(x,0) > 0, \quad x \in I_i,\ i = 1,2,3, \qquad u_0(t) > 0, \quad t \in [0,T], \tag{3.3}$$

then we have

$$u_i(x,t) > 0, \quad (x,t) \in \overline{D}_i,\ i = 1,2,3.$$

Proof. The proof is similar to Lemma 2.5 in the paper [8]. □

Lemma 3.3. *Let* $s(t) \in C[0,T], s(t) > l_2 > 0, t \in [0,T]$. *Suppose* $u_i(x,t) \in C_1^2(D_i) \cap C_0^1(\overline{D}_i),\ i = 1,2,3$ *satisfy* (2.1)–(2.10). *If*

$$0 < u_i(x,0) \le u_M, \quad x \in I_i,\ i = 1,2,3, \qquad 0 < u_0(t) \le u_M, \quad t \in [0,T], \tag{3.4}$$

then we have

$$0 < u_i(x,t) \le u_M, \quad (x,t) \in \overline{D}_i,\ i = 1,2,3.$$

Proof. Setting

$$v_i(x,t) = u_i(x,t) - u_M, \quad (x,t) \in \overline{D}_i,\ i = 1,2,3.$$

By Lemma 3.1, for $i = 1,2,3$ and $(x,t) \in \overline{D}_i$, we have

$$\begin{aligned} v_i(x,t) \le \max_{0\le t\le T} \Big\{ & v_1(0,t),\ v_3(s(t),t),\ \max_{x\in I_1} v_1(x,0),\ \max_{x\in I_2} v_2(x,0), \\ & \max_{x\in I_3} v_3(x,0),\ v_1(l_1,t),\ v_2(l_1,t),\ v_2(l_2,t),\ v_3(l_2,t) \Big\}. \end{aligned}$$

From (3.4), we know

$$\begin{cases} v_1(0,t) = u_0(t) - u_M \le 0, \\ v_3(s(t),t) = 0, \\ v_i(x,0) = u_i(x,0) - u_M \le 0. \end{cases}$$

If $v_1(l_1,t), v_2(l_1,t), v_2(l_2,t), v_3(l_2,t) \le 0$ for $0 \le t \le T$, then we have $v_i(x,t) \le 0$ in $\overline{D}_i,\ i = 1,2,3$. Further $u_i(x,t) \le u_M$ and we have already proved the result. Otherwise, there is a minimum time $t_0 \in (0,T]$ and $i \in \{1,2,3\}, j \in \{1,2\}$ such that

$$v_i(l_j,t_0) = m = \max_{t\in[0,T]} \{v_1(l_1,t), v_2(l_1,t), v_2(l_2,t), v_3(l_2,t)\} > 0. \tag{3.5}$$

If $v_1(l_1,t_0) = m$. By Lemma 3.1, we have

$$v_1(x,t) \le v_1(l_1,t_0), \quad (x,t) \in I_1 \times [0,t_0],$$

namely $v_1(x,t)$ attains its maximum over $I_1 \times [0,t_0]$ at (l_1,t_0). If there is a point in $(0,l_1) \times (0,t_0]$ such that $v_1(x,t)$ attains its maximum, then by the strong maximum principle (refer to [10], pp. 54), we have $v_1(x,t) \equiv C$ for $x \in I_1,\ 0 \le t \le t_0$ where C is constant. Otherwise, for all $(x,t) \in (0,l_1) \times (0,t_0]$, we have $v_1(x,t) < m$, by the strong maximum principle (refer to [9], pp. 170), we know $\frac{\partial v_1}{\partial x}(l_1,t_0) > 0$. If $v_1(x,t) \equiv C$, by $v_1(0,t) \le 0$, $v_1(l_1,t_0) > 0$, we can see that there is a contradiction. If $\frac{\partial v_1}{\partial x}(l_1,t_0) > 0$, by

$$\lambda_1 \frac{\partial v_1}{\partial x}(l_1,t_0) = \sigma_1(u_2^4(l_1,t_0) - u_1^4(l_1,t_0)), \tag{3.6}$$

we know $u_2^4(l_1, t_0) > u_1^4(l_1, t_0)$, from Lemma 3.2, we have $u_2(l_1, t_0) > u_1(l_1, t_0)$ and $v_2(l_1, t_0) > v_1(l_1, t_0)$ which has a contradiction with (3.5).

For the case of $v_2(l_1, t_0) = m$, by the same method, we can prove that $v_2(x, t) \equiv C$ for $x \in I_2$, $0 \le t \le t_0$ or $-\frac{\partial v_2}{\partial x}(l_1, t_0) > 0$, where C is constant. If $v_2(x, t) \equiv C$, by $v_2(0, t) \le 0$, $v_2(l_1, t_0) > 0$, we can see that there is a contradiction. If $\frac{\partial v_2}{\partial x}(l_1, t_0) < 0$, by

$$\lambda_1 \frac{\partial v_1}{\partial x}(l_1, t_0) = \lambda_2 \frac{\partial v_2}{\partial x}(l_1, t_0) = \sigma_1(u_2^4(l_1, t_0) - u_1^4(l_1, t_0)), \tag{3.7}$$

we know $u_2^4(l_1, t_0) < u_1^4(l_1, t_0)$, from Lemma 3.2, we have $u_2(l_1, t_0) < u_1(l_1, t_0)$ and $v_2(l_1, t_0) < v_1(l_1, t_0)$ which has a contradiction with (3.5).

For the other cases of $v_2(l_2, t_0) = m$ or $v_3(l_2, t_0) = m$, the proofs are similar. Thus, the proof is completed. □

Under a physically reasonable condition, the following theorem give the uniqueness of moving boundary for problem (2.1)–(2.10).

Theorem 3.4 (*Uniqueness*). *For $j = 1, 2$, we set $D_3^j = \{(x, t) \mid l_2 < x < s_j(t),\ 0 < t < T\}$, where $s_j(t) \in C^2[0, T]$, $s_j(t) > l_2$ for $0 \le t \le T$. Let $u_i^j(x, t) \in C_1^2(D_i) \cap C_0^1(\overline{D}_i)$, $i = 1, 2$ and $u_3^j(x, t) \in C_1^2(D_3^j) \cap C_0^1(\overline{D_3^j})$ satisfy equations*

$$\frac{\partial u_i^j}{\partial t}(x, t) = a_i^2 \frac{\partial^2 u_i^j}{\partial x^2}(x, t), \quad \text{in } D_i,\ i = 1, 2, \tag{3.8}$$

$$\frac{\partial u_3^j}{\partial t}(x, t) = a_3^2 \frac{\partial^2 u_3^j}{\partial x^2}(x, t), \quad \text{in } D_3^j, \tag{3.9}$$

with boundary conditions

$$u_1^j(0, t) = u_0(t), \quad 0 \le t \le T, \tag{3.10}$$

$$u_3^j(s_j(t), t) = u_M, \quad 0 \le t \le T, \tag{3.11}$$

and interface conditions

$$\lambda_1 \frac{\partial u_1^j}{\partial x}(l_1, t) = \sigma_1 \left((u_2^j(l_1, t))^4 - (u_1^j(l_1, t))^4 \right), \quad 0 \le t \le T, \tag{3.12}$$

$$\lambda_1 \frac{\partial u_1^j}{\partial x}(l_1, t) = \lambda_2 \frac{\partial u_2^j}{\partial x}(l_1, t), \quad 0 \le t \le T, \tag{3.13}$$

$$\lambda_2 \frac{\partial u_2^j}{\partial x}(l_2, t) = \sigma_2 \left((u_3^j(l_2, t))^4 - (u_2^j(l_2, t))^4 \right), \quad 0 \le t \le T, \tag{3.14}$$

$$\lambda_2 \frac{\partial u_2^j}{\partial x}(l_2, t) = \lambda_3 \frac{\partial u_3^j}{\partial x}(l_2, t), \quad 0 \le t \le T, \tag{3.15}$$

where $j = 1, 2$. We assume that

$$0 < u_i^j(x, 0) \le u_M, \quad x \in I_i, \tag{3.16}$$

and

$$0 < u_0(t) \le u_M, \quad 0 < t < T \text{ and } u_0(t) \not\equiv u_M. \tag{3.17}$$

If there exist $t_1, t_2 \in (0, T)$ such that

$$\frac{\partial u_1^1}{\partial x}(0, t) = \frac{\partial u_1^2}{\partial x}(0, t), \quad t_1 < t < t_2,$$

then $s_1(t) = s_2(t)$, $0 \le t \le T$.

Proof. Let $s_1 \not\equiv s_2$ in $(0, T)$, then there exists $t_0 \in (0, T)$ such that $s_1(t_0) \ne s_2(t_0)$. Without loss of generality, we may assume that $x_0 = s_2(t_0) < s_1(t_0)$.

By the unique continuation property for a heat equation (e.g., Isakov [11], Chapter 3) for $u_1^1 - u_1^2$, we see that $u_1^1 = u_1^2$ on $\overline{D}_1$. Meanwhile, $u_1^1(l_1, t) = u_1^2(l_1, t)$, $\partial_x u_1^1(l_1, t) = \partial_x u_1^2(l_1, t)$. According to the interface conditions (3.12)–(3.13) and $u_2^j(l_1, t) > 0$ for $j = 1, 2$, we have $u_2^1(l_1, t) = u_2^2(l_1, t)$, $\partial_x u_2^1(l_1, t) = \partial_x u_2^2(l_1, t)$.

Similarly, we can obtain $u_2^1(l_2, t) = u_2^2(l_2, t)$, $\partial_x u_2^1(l_2, t) = \partial_x u_2^2(l_2, t)$ and $u_3^1(l_2, t) = u_3^2(l_2, t)$, $\partial_x u_3^1(l_2, t) = \partial_x u_3^2(l_2, t)$.

By the unique continuation property for $u_3^1 - u_3^2$, we also obtain that $u_3^1 = u_3^2$ on $\overline{D_3^1 \cap D_3^2}$. In particular, by (3.11), we have $u_3^1(x_0, t_0) = u_3^2(x_0, t_0) = u_M$.

By Lemma 3.3, we know $u_3^1(x, y)$ attains its maximum over $\overline{D_3^1}$ at (x_0, t_0), by the strong maximum principle (refer to [10], pp. 54), we know $u_3^1(x, t) \equiv u_M$ on $\overline{D}^1_{3t_0} = \{l_2 \le x \le s_1(t), 0 \le t \le t_0\}$. By the unique continuation, we have $u_3^1(x, t) \equiv u_M$ on D_3^1. Therefore, $u_3^1(l_2, t) = u_M$ and $\partial_x u_3^1(l_2, t) = 0$. According to the interface condition (3.14) and (3.15), we have $u_2^1(l_2, t) = u_M$ and $\partial_x u_2^1(l_2, t) = 0$, then by the unique continuation again, we obtain $u_2^1 \equiv u_M$ in D_2.

Similarly, we can obtain $u_1^1 \equiv u_M$ in D_1, from condition (3.17), we can see that there is a contradiction.

Thus, $s_1(t) = s_2(t)$ for $t \in (0, T)$, by the continuity of $s_1(t)$ and $s_2(t)$ on $[0, T]$, the proof is completed. □

References

[1] K. Bryan, L. Caudill, Reconstruction of an unknown boundary portion from Cauchy data in *n* dimensions, Inverse Problems 21 (1) (2005) 239–255.

[2] B. Canuto, E. Rosset, S. Vessella, Quantitative estimates of unique continuation for parabolic equations and inverse initial-boundary value problems with unknown boundaries, Trans. Amer. Math. Soc. 354 (2) (2002) 491–535 (electronic).

[3] S. Vessella, Stability estimates in an inverse problem for a three-dimensional heat equation, SIAM J. Math. Anal. 28 (6) (1997) 1354–1370.

[4] A.El. Badia, F. Moutazaim, A one-phase inverse Stefan problem, Inverse Problems 15 (6) (1999) 1507–1522.

[5] T.P. Fredman, A boundary identification method for an inverse heat conduction problem with an application in ironmaking, Heat Mass Transfer 41 (2004) 95–103.

[6] J. Liu, B. Guerrier, A comparative study of domain embedding methods for regularized solutions of inverse Stefan problems, Internat. J. Numer. Methods Engrg. 40 (19) (1997) 3579–3600.

[7] P. Manselli, S. Vessella, On continuous dependence, on noncharacteristic Cauchy data, for level lines of solutions of the heat equation, Forum Math. 3 (5) (1991) 513–521.

[8] G.F. Yang, M. Yamamoto, J. Cheng, Heat transfer in composite materials with Stefan–Boltzmann interface conditions, Math. Methods Appl. Sci. 31 (11) (2008) 1297–1314.

[9] M.H. Protter, H.F. Weinberger, Maximum Principles in Differential Equations, Springer-Verlag, New York, 1984, Corrected reprint of the 1967 original.

[10] L.C. Evans, Partial Differential Equations, in: Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 1998.

[11] V. Isakov, Inverse Problems for Partial Differential Equations, in: Applied Mathematical Sciences, vol. 127, Springer-Verlag, New York, 1998.