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A Finite Element Approximation of the Stokes Equations

Samy Alami[1], Abdeslam Elakkad[2], Jawad El-Mekaoui[1], Ahmed Elkhalfi[1], Mohammed El Hammoumi[1]
[1]*Laboratoire Génie Mécanique, Faculté des Sciences et Techniques, Fès, Maroc*
[2]*Mathématiques, Centre de Formation des Instituteurs Sefrou, Sefrou, Maroc*
E-mail: *elakkadabdeslam@yahoo.fr*

Abstract

In this work, a numerical solution of the incompressible Stokes equations is proposed. The method suggested is based on an algorithm of discretization by the unstable of $Q_1 - P_0$ velocity/pressure finite element approximation. It is shown that the inf-sup stability constant is $O(h)$ in two dimensions and $O(h^2)$ in three dimensions. The basic tool in the analysis is the method of modified equations which is applied to finite difference representations of the underlying finite element equations. In order to evaluate the performance of the method, the numerical results are compared with some previously published works or with others coming from commercial code like Adina system.

1. Introduction

It is universally recognized that discretization schemes for Stokes and Navier-Stokes equations are subject to an inf-sup or div-stability condition [1]. The stability requirement is manifested in practical computations by the predominance of staggered grid finite volume discretizations, and the existence of unnatural velocity-pressure finite element combinations. These typically involve velocity bubble functions, or else have a macro-element definition of the velocity field.

A number of stabilization methods for inf-sup unstable approximations have been developed during the last three decades. These methods can be classified into two kinds. The first one is residual based stabilization, e.g. the absolutely stabilized method introduced by Douglas and Wang [2] and the Galerkin least square methods introduced by Franca and Hughes [3]. The other kind consists of pressure stabilized methods, and includes the global pressure jump stabilized method of Hughes and Franca [4].

In this paper the low order conforming finite element methods like $Q_1 - P_0$ (trilinear/bilinear velocity with constant pressure) for incompressible flow problems are characterised.

The $Q_1 - P_0$ finite element method is particularly controversial [1]. Despite being damned by theoreticians after the discovery of "weakly singular" modes [5,6], $Q_1 - P_0$ is widely used in practice. For the first time a clean analysis of the instability mechanism is presented.

Section 2 presents the model problem used in this paper. The discretization by finite elements is described in Section 3. Numerical experiments carried out within the framework of this publication and their comparisons with other results are shown in Section 4.

2. Governing Equations

We consider the Stokes equations for the flow of an incompressible viscous fluid

$$-v\nabla^2 \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega$$
$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega \qquad (1)$$
$$\boldsymbol{u} = \boldsymbol{g} \quad \text{on } \Gamma$$

where $\Omega = (0, 1)^2$ denotes the unit square with boundary Γ.

v is a given constant called the kinematic viscosity, u is the fluid velocity, p is the pressure field and ∇ is the gradient operator.

The geometry of the domain and boundary condition are shown in **Figure 1**.

3. Finite Element Approximation

In this section we introduce the finite element approximation of the Stokes equations in two dimensions, and we proceed to describe a means of estimating the relevant constant in the inf-sup condition. An alternative
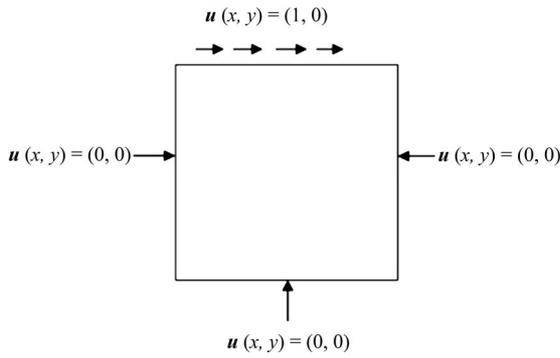
Figure 1. Geometry of the domain and boundary condition for the velocity u.

Figure 2. Q_1-P_0 element (● two velocity components; ○ pressure).

formulation is discussed at the end of this section. Assuming a grid of $n \times n$ square elements K, so that h = 1/n, an approximation based on the Q_1-P_0 element uses function spaces

$$X_h = \left\{ \boldsymbol{u} \in H^1(\Omega)^2 \, \big| \boldsymbol{u}_{|K} \in Q_1(K), \ \forall K \in \Omega \right\}, \quad (2)$$

$$W_h = \left\{ p \in L^2(\Omega) \, \big| p_{|K} \in P_0(K), \ \forall K \in \Omega \right\}, \quad (3)$$

for velocity and pressure, respectively, where $Q_1(K)$ is the space of bilinear functions and $P_0(K)$ is the space of constant functions on K. The nodal positions of the Q_1-P_0 element are illustrated in **Figure 2**. Denoting the velocity space satisfying homogeneous boundary conditions by $X_{h,0} = X_h \cap H_0^1(\Omega)$, the finite element formulation is: find $\boldsymbol{u}_h \in X_h$ which interpolate the data $\boldsymbol{u}_h = \boldsymbol{g}$ at boundary nodes and $p_h \in W_h$ such that

$$\left(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}\right) + \left(\nabla p_h, \boldsymbol{v}\right) = \left(\boldsymbol{f}, \boldsymbol{v}\right) \text{ for all } \boldsymbol{v} \in X_{h,0}, \quad (4)$$
$$-\left(q, \nabla \cdot \boldsymbol{u}_h\right) = 0 \text{ for all } q \in W_h.$$

with standard interpolating bases for the spaces X_h and W_h, this leads to the matrix system

$$\begin{pmatrix} A & 0 & B_x^t \\ 0 & A & B_y^t \\ B_x & B_y & 0 \end{pmatrix} \begin{pmatrix} U \\ V \\ P \end{pmatrix} = \begin{pmatrix} F_x \\ F_y \\ G \end{pmatrix}, \quad (5)$$

where $U, V \in IR^{n_u}$ contain the nodal values of the x and y components of the approximation to $\boldsymbol{u}_h$ at the internal vertices, and $P \in IR^{n_p}$ contains the element centroid values of the pressure approximation. The matrix A is of order $n_u \times n_u$, while both B_x^t and B_y^t are of order $n_u \times n_p$, with $n_u = (n-1)^2$ and $n_p = n^2$.

The satisfaction of the inf-sup condition is dependent on the eigenvalues σ of

$$BK^{-1}B^t P = \sigma N P, \quad (6)$$

where $N = h^2 I$ is the $n_p \times n_p$ mass matrix corresponding to the pressure space [7], $B = \begin{bmatrix} B_x & B_y \end{bmatrix}$ represents

the discrete divergence operator and K is the vector Laplacian

$$K = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \quad (7)$$

Since the discrete velocity field is specified everywhere on the boundary, the discrete Stokes operator has a two-dimensional null space spanned by the hydrostatic and chequerboard pressure modes [8]. Consequently, (6) has only $n^2 - 2$ non-zero eigenvalues and we can order the eigenvalues so that

$$0 = \sigma_1 = \sigma_2 < \sigma_3 \le \sigma_4 \le \cdots \le \sigma_{n^2}. \quad (8)$$

It may also be shown [7], that the nonzero eigenvalues of the "dual" problem

$$B^t N^{-1} B \begin{pmatrix} U \\ V \end{pmatrix} = \sigma K \begin{pmatrix} U \\ V \end{pmatrix} \quad (9)$$

coincide with those of (6). We shall find it more convenient to work with the related eigenproblem

$$\begin{pmatrix} A & 0 & B_x^t \\ 0 & A & B_y^t \\ B_x & B_y & 0 \end{pmatrix} \begin{pmatrix} U \\ V \\ P \end{pmatrix} = \lambda \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & N \end{pmatrix} \begin{pmatrix} U \\ V \\ P \end{pmatrix} \quad (10)$$

which reduces to (6) on elimination of U and V and to (9) on elimination of P. This system has an eigenvalue $\lambda = 1$ of multiplicity equal to the dimension of the null space of the discrete divergence operator, that is $(n-2)^2$, and the remaining $2(n^2 - 2)$ eigenvalues are generated by the quadratic equation $\lambda(\lambda - 1) = \sigma_j$ for $j = 3 \cdots n^2$:

$$\lambda_j^- = \frac{1 - \sqrt{1 + 4\sigma_j}}{2} \le 0, \text{ and } \lambda_j^+ = \frac{1 + \sqrt{1 + 4\sigma_j}}{2} \ge 1. (11)$$

Applying the analysis of Brezzi and Fortin [1] or Malkus [7] the inf-sup stability of the system (4) (or, equivalently, (5)) is determined by the square root of the smallest non zero eigenvalue of (6) that is

$$\sqrt{\sigma_3} = \sqrt{\lambda_3^-\left(\lambda_3^- - 1\right)} \quad (12)$$

and we shall, accordingly, refer to λ_3^- as the critical eigenvalue of (10) and denote it by λ^*. Experimental evidence reported in [2] suggests that $\sigma_3 \to 0$ as $h \to 0$ indicating that Q_1-P_0 is not div-stable in general. To in-

vestigate this issue further we write the constituent equations of (10) in finite difference form and then appeal to the method of modified equations. This will allow us to establish the precise behavior of σ_3 for small values of h.

The x and y components of velocity at the grid point (lh, mh), l, $m = 1, 2, \cdots,$ n – 1 are labeled $U_{l,m}, V_{l,m}$, respectively. The pressure is defined at element centroids $\left(\left(l+\frac{1}{2}\right)h, \left(m+\frac{1}{2}\right)h\right)$, l, $m = 0, 1 \cdots,$ n – 1 and is consequently labeled $P_{l+\frac{1}{2},m+\frac{1}{2}}$. The system of Equations (10) (each divided by h^2) may then be expressed as

$$-(1-\lambda)\nabla_h^2 U_{l,m} + h^{-1}\delta_x\mu_y P_{l,m} = 0$$
$$-(1-\lambda)\nabla_h^2 V_{l,m} + h^{-1}\mu_x\delta_y P_{l,m} = 0 \qquad (13)$$
$$-h^{-1}\delta_x\mu_y P_{l+\frac{1}{2},m+\frac{1}{2}} - h^{-1}\mu_x\delta_y V_{l+\frac{1}{2},m+\frac{1}{2}} = \lambda P_{l+\frac{1}{2},m+\frac{1}{2}}$$

where

$$\delta_x P_{l,m} = P_{l+\frac{1}{2},m} - P_{l-\frac{1}{2},m}, \quad \mu_x P_{l,m} = \frac{1}{2}\left(P_{l+\frac{1}{2},m} + P_{l-\frac{1}{2},m}\right), (14)$$

are the usual central difference and averaging operators, respectively, and ∇_h^2 denotes the discrete Laplacian generated by bilinear elements

$$\nabla_h^2 U_{l,m} = h^{-2}\left(\delta_x^2 + \delta_y^2 + \frac{1}{3}\delta_x^2\delta_y^2\right)U_{l,m} \qquad (15)$$

Figure 2 shows the Q_1-P_0 element and **Figure 3** shows the pressure and the components of the velocity at the grid point.

Associated with (13) we have the boundary conditions $U_{l,m} = V_{l,m} = 0$ at nodes lying on the boundary Γ. The system (13) is

$$-\nabla^2\boldsymbol{u} + \nabla p = -\lambda\nabla^2\boldsymbol{u} \quad \text{in } \Omega$$
$$-\nabla.\boldsymbol{u} = \lambda p \quad \text{in } \Omega, \qquad (16)$$
$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma$$
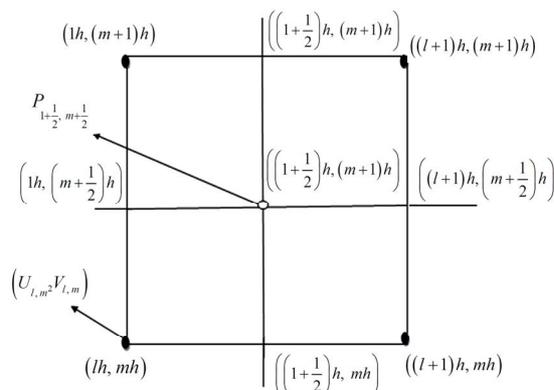
Figure 3. Pressure and the components of the velocity at the grid point.

which has one eigenvalue $\lambda = 0$ corresponding to the hydrostatic mode $\boldsymbol{u} = \boldsymbol{0}$, p = constant, an eigenvalue $\lambda = 1$ of infinite multiplicity with corresponding eigenfunctions satisfying $\nabla\cdot\boldsymbol{u} = 0$, p = 0, and a pair of eigenvalues $\lambda = \frac{1\pm\sqrt{5}}{2}$ of infinite multiplicity corresponding to $\sigma = 1$ and having irrotational eigenvectors $(\nabla\times\boldsymbol{u} = 0)$.

In order to study the behavior of the critical eigenvalue λ^*, we assume that the corresponding eigenfunctions are highly oscillatory and have (smooth) envelopes $(\bar{U},\bar{V},\bar{P})$ defined by

$$U_{l,m} = h^2(-1)^{l+m}\bar{U}_{l,m}, \quad V_{l,m} = h^2(-1)^{l+m}\bar{V}_{l,m},$$
$$P_{l+\frac{1}{2},m+\frac{1}{2}} = (-1)^{l+m+1}\bar{P}_{l+\frac{1}{2},m+\frac{1}{2}}. \qquad (17)$$

It then follows that

$$\delta_x^2 U_{l,m} = h^2(-1)^{l+m+1}\left(4+\delta_x^2\right)\bar{U}_{l,m} \qquad (18)$$

So that

$$\nabla_h^2 U_{l,m} = \frac{1}{3}(-1)^{l+m+1}\left(8-\delta_x^2-\delta_y^2-\delta_x^2\delta_y^2\right)\bar{U}_{l,m}. \qquad (19)$$

Moreover,

$$\delta_x\mu_y P_{l,m} = (-1)^{l+m+1}\mu_x\delta_y\bar{P}_{l,m} \qquad (20)$$

$$\mu_x\delta_y P_{l,m} = (-1)^{l+m+1}\delta_x\mu_y\bar{P}_{l,m} \qquad (21)$$

$$\delta_x\mu_y U_{l+\frac{1}{2},m+\frac{1}{2}} = h^2(-1)^{l+m}\mu_x\delta_y\bar{U}_{l+\frac{1}{2},m+\frac{1}{2}} \qquad (22)$$

$$\mu_x\delta_y V_{l+\frac{1}{2},m+\frac{1}{2}} = h^2(-1)^{l+m}\delta_x\mu_y\bar{V}_{l+\frac{1}{2},m+\frac{1}{2}} \qquad (23)$$

When these relations are used to change dependent variables in (13) from (U,V,P) to $(\bar{U},\bar{V},\bar{P})$ we find that

$$\frac{1}{3}(1-h^2\mu)\left[8-\delta_x^2-\delta_y^2-\delta_x^2\delta_y^2\right]\bar{U}_{l,m} - h^{-1}\delta_y\mu_x\bar{P}_{l,m} = 0,$$

$$\frac{1}{3}(1-h^2\mu)\left[8-\delta_x^2-\delta_y^2-\delta_x^2\delta_y^2\right]\bar{V}_{l,m} - h^{-1}\mu_y\delta_x\bar{P}_{l,m} = 0, (24)$$

$$h^{-1}\mu_x\delta_y\bar{U}_{l+\frac{1}{2},m+\frac{1}{2}} + h^{-1}\delta_x\mu_y\bar{V}_{l+\frac{1}{2},m+\frac{1}{2}} = \mu\bar{P}_{l+\frac{1}{2},m+\frac{1}{2}},$$

where $\mu = \frac{\lambda}{h^2}$, together with homogeneous Dirichlet boundary conditions on $\bar{U}$ and $\bar{V}$. Denoting the Q_1 basis function at node (lh,mh) by $\varphi_{l,m}$, and supposing that $\boldsymbol{u}_h = [u_h, v_h] \in X_h$ has nodal values $[U_{l,m},V_{l,m}]$, then, since $(\nabla\varphi_{l,m},\nabla u_h) = h^2\nabla_h^2 U_{l,m}$ and

$(\varphi_{l,m},u_h) = h^2\left(1+\frac{1}{6}\delta_x^2\right)\left(1+\frac{1}{6}\delta_y^2\right)U_{l,m}$, it is easy to see that (24) is the $Q_1 - P_0$ discretization of the modified system

$$\frac{1}{3}\left(1-h^2\mu\right)\left[8\bar{u}-\frac{7}{3}h^2\nabla^2\bar{u}-\frac{4}{9}h^2\bar{u}_{xxyy}\right]-\bar{p}_y=0 \quad \text{in } \Omega \ ,$$

$$\frac{1}{3}\left(1-h^2\mu\right)\left[8\bar{v}-\frac{7}{3}h^2\nabla^2\bar{v}-\frac{4}{9}h^2\bar{v}_{xxyy}\right]-\bar{p}_x=0 \quad \text{in } \Omega \ ,$$

$$\bar{u}_y+\bar{v}_x \ = \mu\bar{p} \quad \text{in } \Omega \ ,$$

$$\bar{u}=\bar{v}=0 \quad \text{on } \Gamma \ .$$

$$(25)$$

In the limit $h \to 0$ we obtain the reduced problem

$$\frac{8}{3}\bar{u}-\bar{p}_y=0 \qquad \text{in } \Omega \ ,$$

$$\frac{8}{3}\bar{v}-\bar{p}_x=0 \qquad \text{in } \Omega \ ,$$

$$\bar{u}_y+\bar{v}_x \ = \mu\bar{p} \quad \text{in } \Omega \ ,$$

$$\bar{u}=\bar{v}=0 \qquad \text{in } \Gamma \ .$$

$$(26)$$

This implies that

$$\nabla^2\varphi-\frac{8}{3}\mu\varphi=0 \ \text{ in } \Omega \ . \tag{27}$$

For each of the dependent variables $\varphi=\bar{u},\bar{v},\bar{p}$. This, being a second order elliptic eigenvalue problem, requires only one boundary condition whereas the system (26) contains two. The Equations (25) therefore represent a singularly perturbed system [9] whose solutions will, in general, contain boundary layers. Bearing this in mind, we may identify the smallest non zero eigenvalue of (26) as

$$\mu=-\frac{3}{8}\pi^2+O\left(h\right) \tag{28}$$

associated with which there are two eigenfunctions, having outer expansions

$$\begin{pmatrix}\bar{u}_1\\\bar{v}_1\\\bar{p}_1\end{pmatrix}=\begin{pmatrix}\mu\sin\left(\pi y\right)\\0\\\pi\cos\left(\pi y\right)\end{pmatrix} \ , \quad \begin{pmatrix}\bar{u}_2\\\bar{v}_2\\\bar{p}_2\end{pmatrix}=\begin{pmatrix}0\\\mu\sin\left(\pi x\right)\\\pi\cos\left(\pi x\right)\end{pmatrix}. \tag{29}$$

It is important to note that since $\bar{u}_1=0$ on Γ, it exhibits boundary layers of width $O(h)$ along the vertical boundaries $x=0,1$. Similarly, $\bar{v}_2$ has boundary layers of width $O(h)$ along the horizontal boundaries $y=0,1$. We also note that, for this modified system, a "pressure gradient" in the y-direction induces a "flow" in the x-direction and vice-versa. We summarise in the form of a theorem.

Theorem 1: The critical eigenvalue of (10) is given by $\lambda^*=-\frac{3}{8}\pi^2h^2+O\left(h^3\right)$ and the inf-sup constant for Q_1-P_0 elements is given by

$$\sqrt{\sigma_3}=\sqrt{\frac{3}{8}}\pi h+O\left(h^2\right). \tag{30}$$

The eigenspace corresponding to λ^* has dimension two and is spanned by mutually orthogonal eigenvectors given, approximately, by

$$\begin{pmatrix}U_{l,m}\\V_{l,m}\\P_{l+\frac{1}{2},m+\frac{1}{2}}\end{pmatrix}=\left(-1\right)^{l+m+1}\begin{pmatrix}\frac{3}{8}\pi^2h^2\sin\left(\pi mh\right)\\0\\\pi\cos\left(\pi\left(m+\frac{1}{2}\right)h\right)\end{pmatrix}, \tag{31}$$

and

$$\begin{pmatrix}U_{l,m}\\V_{l,m}\\P_{l+\frac{1}{2},m+\frac{1}{2}}\end{pmatrix}=\left(-1\right)^{l+m+1}\begin{pmatrix}0\\\frac{3}{8}\pi^2h^2\sin\left(\pi lh\right)\\\pi\cos\left(\pi\left(l+\frac{1}{2}\right)h\right)\end{pmatrix}, \tag{32}$$

at internal nodes of the domain and $U_{l,m}=V_{l,m}=0$ at boundary nodes. The amplitudes of the velocity and pressure contributions to these critical eigenvectors differ by a factor $O\left(h^2\right)$, which explains why, in computations, the instabilities manifest themselves most strongly in the pressure field. This will be explored further in the next section in connection with the model lid driven cavity problem.

An alternative Stokes formulation to (1) is the stress divergence form [1]:

$$-\nabla^2\boldsymbol{u}-\nabla\left(\nabla\boldsymbol{u}\right)+\nabla p=\boldsymbol{f} \quad \text{in } \Omega$$

$$\nabla\cdot\boldsymbol{u}=0 \quad \text{in } \Omega \tag{33}$$

$$\boldsymbol{u}=\boldsymbol{g} \quad \text{on } \Gamma$$

which after discretisation gives a matrix system which is slightly different to (5) above. If we discretise (33) using Q_1-P_0 and follow the procedure described by (17) for changing variables, then it is easily shown that the resulting reduced problem is also of the form (26) except that the factor $\frac{8}{3}$ is replaced by 4. In this case we have the result stated below.

Theorem 2: The inf-sup constant for Q_1-P_0 approximation of (33) is given by

$$\sqrt{\sigma_3}=\frac{1}{2}\pi h+O\left(h^2\right). \tag{34}$$

The corresponding eigenspace has dimension two and is spanned by mutually orthogonal eigenvectors given, approximately, by (31) and (32).

4. Numerical Simulations

In this section, some numerical results of calculations with finite element method and ADINA system will be

presented. Using our solver, we run the traditional test problem driven cavity flow [10-13].

We consider the critical eigenvalue of Theorem 1. The numerically computed eigenvalues σ_j are presented in **Table 1**. The smallest eigenvalues are clearly tending to zero like $O(h^2)$. The largest eigenvalue is converging to an asymptotic limit of unity. Note also that for "unstable" eigenvalues: $\sigma_j = O(h^2)$ implies that

$$\lambda_j^- = -\sigma_j + O\left(h^3\right) \text{ and } \lambda_j^+ = 1 + \sigma_j + O\left(h^3\right) \quad (35)$$

Suggesting that $\lambda^* \to -\sigma_3$ as $h \to 0$

Figures 4 and **5** show, respectively, the contours of pressure component of the first and the second unstable eigenvectors corresponding to λ^*, whereas **Figures 6** and **7** show contours of pressure component of the unstable eigenvector corresponding to σ_5 and those cor-

responding to σ_{n^2}.

The profiles of the *u*-velocity component along the vertical center line and the *v*-velocity component along the horizontal center line are shown in **Figure 8**. In this figure, we have the excellent agreement between the computed results and the results computed with ADINA system.

Table 1. Behavior of the eigenvalues σ_j satisfying (6).

Grid	$\sigma_3 = \sigma_4$	σ_5	σ_{n^2}
8×8	4.65×10^{-2}	7.33×10^{-2}	0.975
16×16	1.32×10^{-2}	2.39×10^{-2}	0.994
32×32	3.45×10^{-3}	6.62×10^{-3}	0.998
64×64	8.85×10^{-4}	1.73×10^{-3}	0.999
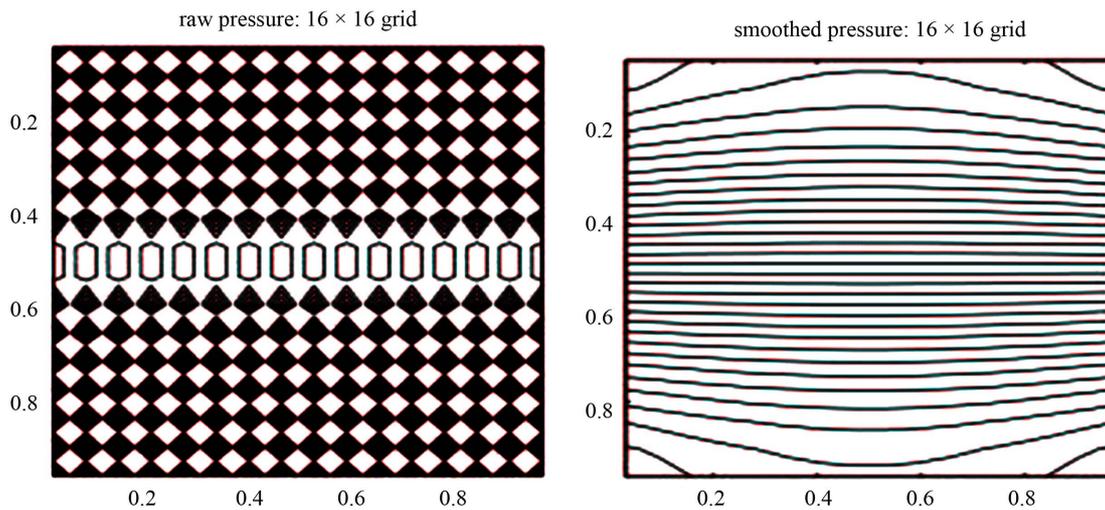
Figure 4. Contours of pressure component of the first unstable eigenvector corresponding to λ^*.
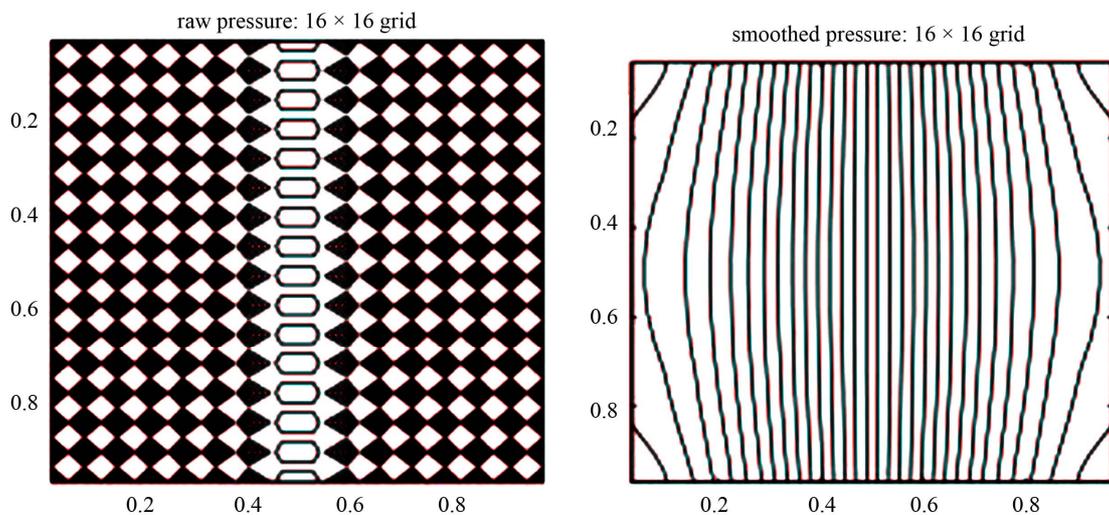
Figure 5. Contours of pressure component of the second unstable eigenvector corresponding to λ^*.

raw pressure: 16 × 16 grid   smoothed pressure: 16 × 16 grid
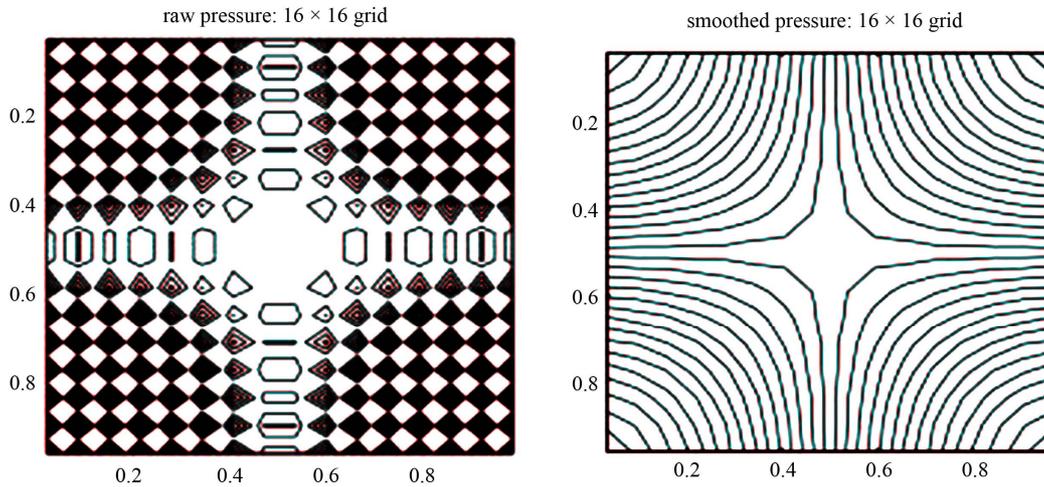
Figure 6. Contours of pressure component of the unstable eigenvector corresponding to σ_s .

raw pressure: 16 × 16 grid
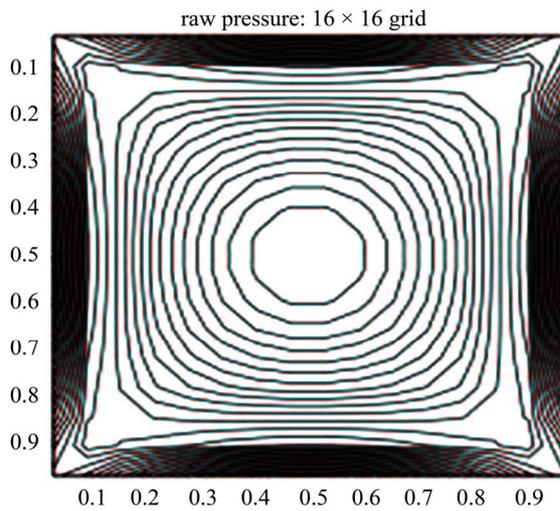
Figure 7. Contours of pressure component of the unstable eigenvector corresponding to σ_{n2} .

5. Conclusions

We were interested in this work in the numerical solution for two dimensional differential equations model steady incompressible flow. It includes algorithms for discretization by finite element methods. For the test of driven-cavity flow, the particles in the body of the fluid move in a circular trajectory.

Our results agree with Adina System.

Numerical results are presented to see the performance of the method, and seem to be interesting by comparing them with other recent results.

6. Acknowledgments

(a)

(b)
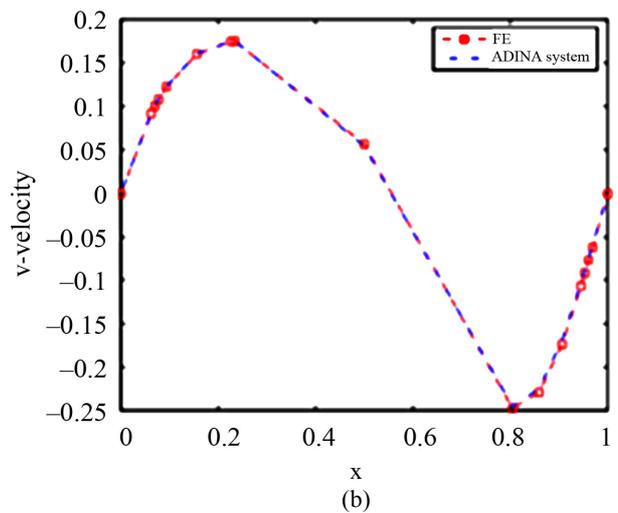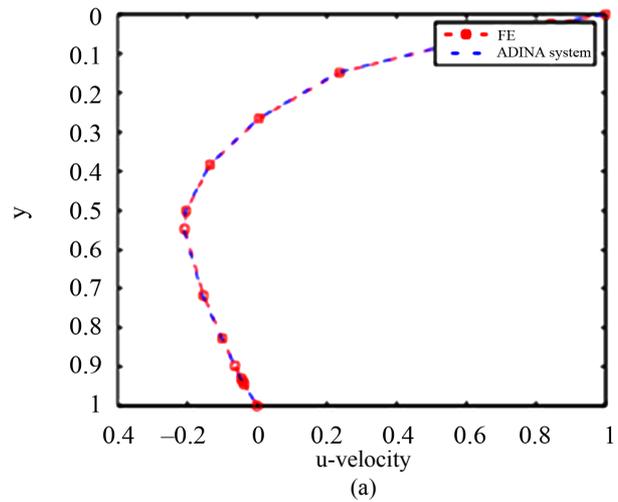
Figure 8. The velocity component *u* at vertical center line (a), and the velocity component *v* at horizontal center line (b) with a 129 × 129 grid.

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