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Ranks and Subdegrees of External Direct Product of $C_n \times D_r$ Acting on $X \times Y$

Felix Mutinda Mutua

Department of Pure and Applied Mathematics
Jomo Kenyatta University of Agriculture and Technology (JKUAT)
P.O. Box 62000 - 00200, Nairobi, Kenya
Corresponding author

Peter Waweru Kamaku

Department of Pure and Applied Mathematics
Jomo Kenyatta University of Agriculture and Technology (JKUAT)
P.O. Box 62000 - 00200, Nairobi, Kenya

Patrick Mwangi Kimani

Department of Mathematics and Statistics
University of Kbianga (UoK)
P.O. Box 2030 - 20200 Kericho, Kenya

Abstract

In this paper, transitivity, ranks and subdegrees of the action of external direct product of Cyclic and Dihedral group on Cartesian Product of two sets are determined. The action is proved to be transitive. Also, it's established that the rank associated with the action is $n(\frac{r+1}{2})$ and subdegrees are $[1]^{[n]}$ and $[2]^{[n(\frac{r-1}{2})]}$ when r is old. Additionally, the rank of the action for the case where r is even is proved to be $n(\frac{r+2}{2})$ and subdegrees are $[1]^{[2n]}$ and $[2]^{[n(\frac{r-2}{2})]}$.

Keywords: Action, Cyclic group, Dihedral group, External direct product, Cartesian product, Rank, Subdegrees

1 Introduction

The action of $G = G_1 \times G_2 = C_n \times D_r$ on $X \times Y$ is defined by $(g_1, g_2)(x, y) = (g_1x, g_2y)\ \forall\ g_1 \in G_1,\ g_2 \in G_2$ and $x \in X$ and $y \in Y$. Therefore, this paper investigates and establishes some results and properties associated with action of $C_n \times D_r$ on $X \times Y$.

2 Preliminary Results

2.1 Definitions and Theorems

Definition 2.1.1. *Suppose* G *acts on* X*. Then the set* G*-orbit also denoted as* $Orb_G(x)$ *of points* $gx\ \forall\ g \in G$ *is called orbit of* G *containing* x *[1].*

Definition 2.1.2. *Let* G *act on a finite set* X*. Then, the stabilizer of a point* $x \in X$ *in* G *is the set* G_x *of all elements in* $G : gx = x$ *[2].*

Definition 2.1.3. *Let* G *act on defined set* X *transitively. Then suborbits is the* $G_x - orbits$ *on* X*. The number of the* $G_x - orbits$ *on* X *is the rank while the length of each* $G_x - orbit$ *is the subdegree. The symbol* $\Delta(x)$*, denotes* $G_x - orbits$ *[3].*

Definition 2.1.4. *If for every pair* $x_1, x_2 \in X\ \exists\ g \in G : gx_1 = x_2$*. Then this shows the action possesses one orbit hence transitive [4].*

Theorem 2.1.5. *Let* $G = C_n$ *act on set* X*. Then for each* x *in* X $Stab_G(x) = \{e\}$ *[5].*

Theorem 2.1.6. *Suppose* $G = D_r$ *acts on set* X*. Then,* $Stab_G(1) = \{e, (2r)(3r-1)(4r-2)\dots(\frac{r}{2}\frac{r+4}{2})\}$ *when* r *is even and* $Stab_G(1) = \{e, (2r)(3r-1)(4r-2)\dots(\frac{r+1}{2}\frac{r+3}{2})\}$ *when* r *is odd [5].*

Theorem 2.1.7. *Suppose* $G = C_n$ *acts on set* X*. Then, orbits of* $Stab_G(1)$ *on* X *are* $\Delta_0 = \{1\}$, $\Delta_1 = \{2\}$, $\Delta_2 = \{3\}$, $\dots$, $\Delta_i = \{i+1\}$, $\dots$, $\Delta_{n-1} = \{n\}$ *[5].*

Theorem 2.1.8. *Let* $G = D_r$ *act on set* X*. Then, orbits of* $Stab_G(1)$ *on* X *are* $\Delta_0 = \{1\}$, $\Delta_1 = \{2, r\}$, $\dots$, $\Delta_i = \{i+1, r-i+1\}$, $\dots$, $\Delta_{\frac{r}{2}} = \{\frac{r}{2}+1\}$ *when* r *is even, for* r *odd the* $G_1 - orbits$ *are* $\Delta_0 = \{1\}$, $\Delta_1 = \{2, r\}$, $\dots$, $\Delta_i = \{i+1, r-i+1\}$, $\dots$, $\Delta_{\frac{r-1}{2}} = \{\frac{r+1}{2}, \frac{r+3}{2}\}$ *[5].*

Theorem 2.1.9. *Let* $G = G_1 \times G_2$ *act on* $X \times Y$*. Then* $Orb_G(x, y) = Orb_{G_1}(x) \times Orb_{G_2}(y)$ *and* $Stab_G(x, y) = Stab_{G_1}(x) \times Stab_{G_2}(y)$ *for* $(x, y) \in X \times Y$ *[6].*

Theorem 2.1.10. *Orbit-Stabilizer Theorem; If* G *acts on* X *transitively. Then for* $x \in X$, $|G_x| = \frac{|G|}{|Orb_G(x)|}$ *[7].*

3 Main Results

3.1 Transitivity of $C_n \times D_r$ on $X \times Y$

Theorem 3.1.1. *Let $G_1 = C_n$ be a cyclic group generated by $(12345\dots n)$ and $G_2 = D_r$ be a dihedral group of order $2r$. Then, $G = G_1 \times G_2 = C_n \times D_r$ acts transitively on $X \times Y$, where $X = \{1, 2, \dots, n\}$ and $Y = \{1, 2, \dots, r\}$.*

Proof. By Theorem 2.1.5 and 2.1.6, $H_1 = Stab_{G_1}(1) = \{I\}$ and $H_2 = Stab_{G_2}(1)$ a subgroup of order 2 respectively. It follows from Theorem 2.1.9, $H = Stab_G(1,1) = Stab_{G_1}(1) \times Stab_{G_2}(1) = H_1 \times H_2$. Thus, by Theorem 2.1.10, $|Orb_G(1,1)| = \frac{|G|}{|H|} = \frac{2nr}{2} = |nr| = |X \times Y|$. This proves the action is transitive. □

3.2 Ranks and subdegrees of $C_n \times D_r$ on $X \times Y$, where r is odd.

Theorem 3.2.1. *Let $G = C_n \times D_r$ act on $X \times Y$, where $X = \{1, 2, \dots, n\}$ and $Y = \{1, 2, \dots, r\}$. Then the rank is $n(\frac{r+1}{2})$ and subdegrees are $[1]^{[n]}$ and $[2]^{[n(\frac{r-1}{2})]}$.*

Proof. Let $H_1 = Stab_{G_1}(1)$, $H_2 = Stab_{G_2}(1)$ and $H = Stab_G(1,1)$. By Theorem 2.1.9, $H = H_1 \times H_2$. Suppose H_1 acts on X, then by Theorem 2.1.7, the H_1-orbits are $X_0 = \{1\}$, $X_2 = \{2\}$, $X_3 = \{3\}, \dots$, $X_{n-1} = \{n\}$. Similarly, if H_2 acts on Y, by Theorem 2.1.8, H_2-orbits are $Y_0 = \{1\}$, $Y_1 = \{2, r\}$, $Y_2 = \{3, r-1\}$, $Y_3 = \{4, r-2\}$, $\dots$, $Y_{\frac{r-1}{2}} = \{\frac{r+1}{2}, \frac{r+3}{2}\}$. It follows from Theorem 2.1.9, that:

$$\Delta_0 = Orb_H(1,1) = X_0 \times Y_0 = \{1\} \times \{1\} = \{(1,1)\}. \tag{1}$$

$$\Delta_1 = Orb_H(2,1) = X_1 \times Y_0 = \{2\} \times \{1\} = \{(2,1)\}. \tag{2}$$

$$\Delta_2 = Orb_H(3,1) = X_2 \times Y_0 = \{3\} \times \{1\} = \{(3,1)\}. \tag{3}$$

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$$\Delta_{n-1} = Orb_H(n,1) = X_{n-1} \times Y_0 = \{n\} \times \{1\} = \{(n,1)\}. \tag{4}$$

$$\Delta_n = Orb_H(1,2) = X_0 \times Y_1 = \{1\} \times \{2, r\} = \{(1,2), (1,r)\}. \tag{5}$$

$$\Delta_{n+1} = Orb_H(2,2) = X_1 \times Y_1 = \{2\} \times \{2,r\} = \{(2,2),(2,r)\}. \tag{6}$$

$$\Delta_{n+2} = Orb_H(3,2) = X_2 \times Y_1 = \{3\} \times \{2,r\} = \{(3,2),(3,r)\}. \tag{7}$$

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$$\Delta_{2n-1} = Orb_H(n,2) = X_{n-1} \times Y_1 = \{n\} \times \{2,r\} = \{(n,2),(n,r)\}. \tag{8}$$

$$\Delta_{2n} = Orb_H(1,3) = X_0 \times Y_2 = \{1\} \times \{3,r-1\} = \{(1,3),(1,r-1)\}. \tag{9}$$

$$\Delta_{2n+1} = Orb_H(2,3) = X_1 \times Y_2 = \{2\} \times \{3,r-1\} = \{(2,3),(2,r-1)\}. \tag{10}$$

$$\Delta_{2n+2} = Orb_H(3,3) = X_2 \times Y_2 = \{3\} \times \{3,r-1\} = \{(3,3),(3,r-1)\}. \tag{11}$$

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$$\begin{aligned} \Delta_{\alpha=nj+i} = Orb_H(i+1,j+1) &= X_i \times Y_j = \{i+1\} \times \{j+1, r-1\} \\ &= \{(i+1,j+1),(i+1,r-1)\}. \end{aligned} \tag{12}$$

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$$\begin{aligned} \Delta_{n\frac{(r+1)}{2}-1} = Orb_H(n,\frac{r+1}{2}) = X_{n-1} \times Y_{\frac{r-1}{2}} = \{n\} \times \{\frac{r+1}{2},\frac{r+2}{2}\} \\ = \{(n,\frac{r+1}{2}),(n,\frac{r+2}{2})\}. \end{aligned} \tag{13}$$

From the above equations;
$|\Delta_0| = |\Delta_1| = \ldots, = |\Delta_{n-1}| = 1$ and $|\Delta_n| = |\Delta_{n+1}| = \ldots, = |\Delta_{n\frac{(r+1)}{2}-1}| = 2.$

Therefore, the subdegrees are $[1]^{[n]}$ and $[2]^{[n(\frac{r-1}{2})]}$ and the elements in the H-orbits are $n(1) + 2(\frac{nr-n}{2}) = n+(nr-n) = nr = |X \times Y|$. Therefore, the rank is $n(\frac{r+1}{2})$. □

3.3 Ranks and subdegrees of $C_n \times D_r$ on $X \times Y$, where r is even.

Theorem 3.3.1. *Let $G = C_n \times D_r$ act on $X \times Y$, where $X = \{1, 2, \ldots, n\}$ and $Y = \{1, 2, \ldots, r\}$. Then the rank is $n(\frac{r+2}{2})$ and subdegrees are $[1]^{[2n]}$ and $[2]^{[n(\frac{r-2}{2})]}$.*

Proof. Suppose $G = G_1 \times G_2$. Let $H_1 = Stab_{G_1}(1)$, $H_2 = Stab_{G_2}(1)$ and $H = Stab_G(1,1)$. By Theorem 2.1.9, $H = H_1 \times H_2$. Suppose H_1 acts on X, then by Theorem 2.1.7, the H_1-orbits are $X_0 = \{1\}$, $X_2 = \{2\}$, $X_3 = \{3\}$, $\ldots$, $X_{n-1} = \{n\}$. Also, if H_2 acts on Y, by Theorem 2.1.8, H_2-orbits are $Y_0 = \{1\}$, $Y_1 = \{2, r\}$, $Y_2 = \{3, r-1\}$, $Y_3 = \{4, r-2\}$, $\ldots$, $Y_{\frac{r}{2}} = \{\frac{r+2}{2}\}$. It follows from Theorem 2.1.9, that:

$$\Delta_0 = Orb_H(1,1) = X_0 \times Y_0 = \{1\} \times \{1\} = \{(1,1)\}. \tag{14}$$

$$\Delta_1 = Orb_H(2,1) = X_1 \times Y_0 = \{2\} \times \{1\} = \{(2,1)\}. \tag{15}$$

$$\Delta_2 = Orb_H(3,1) = X_2 \times Y_0 = \{3\} \times \{1\} = \{(3,1)\}. \tag{16}$$

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$$\Delta_{n-1} = Orb_H(n,1) = X_{n-1} \times Y_0 = \{n\} \times \{1\} = \{(n,1)\}. \tag{17}$$

$$\Delta_n = Orb_H(1,2) = X_0 \times Y_1 = \{1\} \times \{2, r\} = \{(1,2), (1,r)\}. \tag{18}$$

$$\Delta_{n+1} = Orb_H(2,2) = X_1 \times Y_1 = \{2\} \times \{2, r\} = \{(2,2), (2,r)\}. \tag{19}$$

$$\Delta_{n+2} = Orb_H(3,2) = X_2 \times Y_1 = \{3\} \times \{2, r\} = \{(3,2), (3,r)\}. \tag{20}$$

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$$\Delta_{2n-1} = Orb_H(n,2) = X_{n-1} \times Y_1 = \{n\} \times \{2,r\} = \{(n,2),(n,r)\}. \quad (21)$$

$$\Delta_{2n} = Orb_H(1,3) = X_0 \times Y_2 = \{1\} \times \{3,r-1\} = \{(1,3),(1,r-1)\}. \quad (22)$$

$$\Delta_{2n+1} = Orb_H(2,3) = X_1 \times Y_2 = \{2\} \times \{3,r-1\} = \{(2,3),(2,r-1)\}. \quad (23)$$

$$\Delta_{2n+2} = Orb_H(3,3) = X_2 \times Y_2 = \{3\} \times \{3,r-1\} = \{(3,3),(3,r-1)\}. \quad (24)$$

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$$\Delta_{\alpha=nj+i} = Orb_H(i+1,j+1) = X_i \times Y_j = \{i+1\} \times \{j+1,r-1\} = \{(i+1,j+1),(i+1,r-1)\}. \quad (25)$$

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$$\Delta_{n\frac{(r+2)}{2}-1} = Orb_H(n,\frac{r+2}{2}) = X_{n-1} \times Y_{\frac{r}{2}} = \{n\} \times \{\frac{r+2}{2}\} = \{(n,\frac{r+2}{2})\}. (26)$$

From the above its clear that;
$|\Delta_0| = |\Delta_1| = \ldots, = |\Delta_{n-1}| = 1$, $|\Delta_n| = |\Delta_{n+1}| = \ldots, = |\Delta_{n(\frac{r}{2})-1}| = 2$ and $|\Delta_{n(\frac{r}{2})}| = |\Delta_{n(\frac{r}{2})+1}| = \ldots, = |\Delta_{n\frac{(r+2)}{2}-1}| = 1$.
Therefore, the subdegrees are $[1]^{[2n]}$ and $[2]^{[n(\frac{r-2}{2})]}$ and the elements in the H-orbits are $(1)(2n) + (2)(\frac{nr-2n}{2}) = 2n + (nr-2n) = nr = |X \times Y|$. Thus, the rank is $n(\frac{r+2}{2})$.

□

4 Conclusion

From the properties studied in this research it can be concluded that:

1. The action of $C_n \times D_r$ on $X \times Y$ is transitive.
2. The rank of the action of $C_n \times D_r$ on $X \times Y$ is $n(\frac{r+1}{2})$ when r is odd.
3. The rank of the action of $C_n \times D_r$ on $X \times Y$ is $n(\frac{r+2}{2})$ when r is even.

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