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Rank and Subdegrees of $PGL(2, q)$ Acting Cosets of $PGL(2, e)$ for q an Even Power of e

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Abstract

The action of projective general group on the cosets of its maximal subgroups has been studied. For instance, [9] studied the action of G on the cosets of $PGL(2, e)$ when q is an odd prime power of e. In this paper, we determine the rank and subdegrees of the action of $PGL(2, q)$ on the cosets of its subgroup $PGL(2, e)$ for odd q and an even power of e. We apply the table of marks to achieve this.

Keywords: Rank, Subdegrees, Mark

1 Introduction

Let a group G act transitively on a set X. The orbits of the stabilizer G_α of a point $\alpha \in X$ are called *suborbits* of G on X. The number $R(G)$ of these suborbits is known as the *rank* of G on X and the suborbits length are known as the *subgegrees* of G on X. Rank and subdegrees are independent of the $\alpha \in X$ chosen. Any group G acts transitively on the set of right cosets of any of its subgroup. In this paper the set X is the set of the right cosets of $H = PGL(2, e)$ in $G = PGL(2, q)$ where q is an even power of e. In this paper both q and e represents p^i for some prime p and $i \in \mathbb{Z}^+$. The subgroup $H < PSL(2, q)$ and therefore H is a proper subgroup of G.

2 Preliminary Notes

Theorem 2.1. *[11] Let G be a group acting on set X and $Orb_G(\alpha)$ be an orbit of G containing $\alpha in X$. Then,*

$$|Orb_G(\alpha)| = \frac{|G|}{|G_\alpha|}. \tag{1}$$

Theorem 2.2. *[4] The following are the subgroups of $PGL(2, q)$ for q odd, where $\delta = \begin{cases} 1, & \text{if } q \equiv 1 mod\ 4 \\ -1, & \text{if } q \equiv -1 mod\ 4 \end{cases}$:*

i. 2 conjugacy classes of cyclic subgroups C_2. One class lies in the subgroup $PSL(2, q)$ and consist of $\frac{q(q+\delta)}{2}$ subgroups. The other class consist of $\frac{q(q-\delta)}{2}$ subgroups.

ii. 1 conjugacy class containing $\frac{q(q\mp\delta)}{2}$ conjugate cyclic subgroups C_h $(h > 2)$ for every $h|q \pm \delta$.

iii. 2 conjugacy classes of dihedral subgroups D_4. One class lie in the subgroup $PSL(2, q)$ consisting of $\frac{q(q^2-1)}{24}$ subgroups. The other class consisting of $\frac{q(q^2-1)}{8}$ subgroups.

iv. 2 conjugacy classes of dihedral subgroups D_{2h}, where $h|\frac{q\pm\delta}{2}$ and $h > 2$. One class lie in the subgroup $PSL(2, q)$ and consist of $\frac{q(q^2-1)}{4h}$ subgroups. The other class consist of $\frac{q(q^2-1)}{4h}$ subgroups.

v. 1 conjugacy class of $\frac{q(q^2-1)}{2h}$ dihedral subgroups D_{2d}, where $\frac{q\pm\delta}{h}$ is an odd integer and $h > 2$.

vi. $\frac{q(q^2-1)}{24}$ subgroups A_4, $\frac{q(q^2-1)}{24}$ subgroups S_4 and $\frac{q(q^2-1)}{60}$ subgroups A_5 when $q \equiv -1$ mod 10. There is only one conjugacy class of any of these types of subgroups and all lie in the subgroup $PSL(2, q)$ except for S_4 when $q \equiv -3$ mod 8.

vii. *1 conjugacy class containing $\frac{q(q^2-1)}{e(e^2-1)}$ conjugate $PSL(2,e)$ where q is a power of e.*

viii. *The subgroups $PGL(2,e)$.*

ix. *The elementary abelian groups P_{p^r} of order p^r for every $r = 1, 2, \ldots, f$.*

x. *Semidirect product of the elementary abelian groups P_{p^r} of order p^r for every $r = 1, 2, \ldots, f$ and a cyclic group C_h with $h|(q-1)$ and $h|(p^r-1)$.*

More details on the subgroup structure of $PGL(2,q)$ and $PSL(2,q)$ are also found in [1], [5], [6] and [10].

Definition 2.3. *[2] Let P_G be a permutation representation (transitive or intransitive) of G on X. The mark of the subgroup H of G in P_G is the number of points of X fixed by every permutation of H.*

In case $G(/H_i)$ is a coset representation, the mark of H_j in $G(/H_i)$ denoted by $m(H_j, H_i, G)$ is the number of cosets of H_i in G left fixed by every permutation of H_j.

Definition 2.4. *[7] defined the mark in terms of normalizers of subgroups of a group as; If $H_j \le H_i \le G$ and $H_{j_1}, H_{j_2}, \cdots, H_{j_n}$ is a complete set of conjugacy class representatives of subgroups of G_i that are conjugate to H_j in G, then*

$$m(H_j, Hi, G) = \sum_{k=1}^{n} |N_G(H_{j_k}) : N_{H_i}(H_{j_k})|. \tag{2}$$

In particular when $n = 1$, H_j is conjugate in H_i to all subgroups H_j that are contained in H_i and conjugate to H_j in G and

$$m(H_j, H_i, G) = |N_G(H_j) : N_{H_i}(H_j)|. \tag{3}$$

[See [8].]

Definitions 2.3, and 2.4 are all equivalent by [8].

Definition 2.5. *Let $F_1, F_2, \cdots, F_t$ be a set of representatives of all distinct conjugacy classes of subgroups of H in G, ordered such that $|F_1| \le |F_2| \le \cdots \le |F_t| = |H|$. The table of marks of H is the matrix, $M = (m_{ij})$, where $m_{ij} = m(F_j, F_i, H)$.*

Let Q_i be the number of suborbits $\triangle_j$ on which the action of H is equivalent to its action on the cosets of $F_i (i = 1, 2, \cdots, t)$. The subdegrees of G acting on right cosets H are obtained by computing all Q_i.

Theorem 2.6. *The numbers Q_i satisfy the system of linear equations,*

$$\sum_{i=j}^{t} Q_i m(F_j, F_i, H) = m(F_j, H, G) \tag{4}$$

for each $j = 1, 2, \cdots, t$. *[See [7].]*

3 Main Results

Lemma 3.1. *Suppose $m(F_a, H, G) = m(H, H, G)$, with $1 < a < t$, then $Q_a = 0$. Moreover, if $F_a < F_b$ and $F_b \neq H$, then $Q_b = 0$.*

Proof. Let T be the table of marks of H. All the entries in the last row of T are 1's. That is $m_{t\,j} = 1\ \forall j = 1, ..., t$. By Theorem 2.6, $Q_t = m(H, H, G)$. Also

$$Q_a m_{aa} + Q_{a+1} m_{aa+1} + \cdots + Q_{t-1} m_{at-1} + Q_t = m(F_a, H, G) = m(H, H, G) \quad (5)$$

$$\Rightarrow Q_a m_{aa} + Q_{a+1} m_{aa+1} + \cdots + Q_{t-1} m_{at-1} = 0 \quad (6)$$

But $m_{i\,j} \geq 0,\ Q_j \geq 0\ \ \forall i = 1, ..., t, j = 1, ..., t$ and $m_{a\,a} \neq 0$. It follows that $Q_a = 0$. If $F_a \leq F_b < H$, then $m_{a\,b} \neq 0$. By Equation (6), It follows that $Q_b = 0$. □

The stabilizer of the coset H is H and the stabilizer of the coset Hg for some $g \in G$ is a conjugate subgroup H_0 of H in G. The stabilizer of Hg in H is $H \cap H_0$. If subgroup F_j in G is not an intersection of H and some conjugate H_0 of H in G, then it cannot be a stabilizer of a coset in H. By Theorem 2.6, $Q_j = 0$. Such subgroups of H can be eliminated from the table of marks of H during computation of subdegrees of G acting on the cosets of H. Also, all the subgroups F_j of H such that $Q_j = 0$ can be eliminated by Lemma 3.1.

Theorem 3.2. *Let $G = PGL(2, q)$ act on the cosets of $H = PGL(2, e)$ where q is odd and an even power of e. Then the rank is $\frac{e^5q - e^5 + e^4q - e^3q + e^3 - 4e^2q + 2e^2 + q^3}{e^2(e^2-1)^2}$ and the subdegrees are as in Table 1 with $\beta = \frac{(e^2-q)(e^4+e^3-e^2q+e^2+e-q^2)}{e^2(e^2-1)^2}$.*

Table 1: Subdegrees of $G = PGL(2, q)$ acting on cosets of $H = PGL(2, e)$ with q odd and even power of e

Suborbit length:	1	$\frac{e(e-1)}{2}$	$\frac{e(e+1)}{2}$	$e(e-1)$	e^2-1	$e(e+1)$	$\frac{e(e^2-1)}{2}$	$e(e^2-1)$
No of sub-orbits:	1	1	1	$\frac{q-2e-3}{2(e+1)}$	$\frac{q-e}{e(e-1)}$	$\frac{q-2e+1}{2(e-1)}$	$2\frac{q-e^2}{e^2-1}$	β

Proof. we first determine the subgroups F which may result from intersection of H and a conjugate subgroup H_0 in G.

i. Suppose $F \cong H \cap H_0$ is isomorphic to a cyclic subgroup C_n where $n|e \pm 1$. Then it must be an intersection of two maximal cyclic subgroups of H and H_0 containing C_n. The two subgroups have the same order and hence they intersect either wholly or at identity. Thus $n = 1$ or $e \pm 1$.

ii. Suppose $F \cong H \cap H_0$ is isomorphic to a dihedral subgroup D_{2n} where $n|e\pm1$ with $n \neq p$. Then it must be an intersection of two maximal dihedral subgroups of H and H_0 containing D_{2n}. The subgroup D_{2n} contains n involutions and a cyclic subgroup C_n. Therefore by i. $n = 1, 2$ or $e \pm 1$.

iii. Suppose $F \cong H \cap H_0$ is isomorphic to an Abelian subgroup of order p^r. Then F must be an intersection of two maximal Abelian subgroups of H and H_0 containing F. The two subgroups are of the same order e and therefore intersection is either identity or the whole subgroup. Thus $r = 0$ or m where $e = p^m$.

iv. Suppose $F \cong H \cap H_0$ is isomorphic to a semidirect product of a Abelian group of order p^r and a cyclic subgroup C_n where $n|e-1$. Then it must be an intersection of two maximal semidirect products of the form $P_e \ltimes C_{e-1}$ of H and H_0 containing F. By i. and iii. $F = I$ or $P_e \ltimes C_{e-1}$.

The representatives of the distinct conjugacy classes of H to consinder are; I, $C_2\,(1)$, $C_2\,(2)$, $D_4(1)$, $D_4(2)$, C_{e-1}, C_{e+1}, A_4, A_5, S_4, P_e, $P_e \ltimes C_{e-1}$, $D_{2(e-1)}$, $D_{2(e+1)}$, $PSL(2,p^r)$ and $PGL(2,p^r)$, where $e = p^r$. By Theorem 2.2, the conjugacy class, A_5 exists only when $e \equiv \pm1 \ mod\ 10$. Next we compute the marks of F in $G(/H)$ using Theorem 2.2, and Definition 2.4 and display them in Table 2 where $\epsilon = \begin{cases} 1, & \text{if } q \equiv 1 mod\ 4 \\ -1, & \text{if } q \equiv -1 mod\ 4 \end{cases}$.

(Note : when $e \equiv 1\ mod\ 4$ and $4 \nmid a$, $\frac{q-1}{2(e-1)}$ is odd, when $e \equiv -1\ mod\ 4$ and $4 \nmid a$ $\frac{q-1}{2(e+1)}$ is odd. All the other cases where a is even, $\frac{q-1}{e\pm1}$ is even.)

Eliminating all subgroups with $m\,(F, H, G) = 1$ by use of Theorem 3.1, we are left with the subgroups I, $C_2(1)$, $C_2(2)$, $D_4(1)$, $D_4(2)$, C_{e-1}, C_{e+1}, P_e, $D_{2(e-1)}$ and $D_{2(e+1)}$. Therefore the required table of marks is Table 3 or 4 according to the nature of e.

Table 2: Marks of F in $G(/H)$ where $G = PGL(2, q)$, $H = PGL(2, e)$ with q odd and even power of e

F	$\|N_H(F)\|$	$\|N_G(F)\|$	$m(F, H, G)$
I	$e\,(e^2-1)$	$q\,(q^2-1)$	$\frac{q\left(q^2-1\right)}{e(e^2-1)}$
$C_2(1)$	$2(e-1)$	$2(q-1)$	$\frac{2e(q-1)}{e^2-1}$
$C_2(2)$	$2(e+1)$	$q-1$	$\frac{2e(q-1)}{e^2-1}$
$D_4(1)$	$\frac{24}{2-\epsilon}$	24	4
$D_4(2)$	$\frac{24}{2+\epsilon}$	24	4
C_{e-1}	$2(e-1)$	$2(q-1)$	$\frac{q-1}{e-1}$
C_{e+1}	$2(e+1)$	$2(q+1)$	$\frac{q-1}{e+1}$
A_4	24	24	1
A_5	60	60	1
S_4	24	24	1
P_e	$e(e-1)$	$q(e-1)$	$\frac{q}{e}$
$P_e \ltimes C_{e-1}$	$e(e-1)$	$e(e-1)$	1
$D_{2(e-1)}$	$2(e-1)$	$4(e-1)$	2
$D_{2(e+1)}$	$2(e+1)$	$4(e+1)$	2
$PSL(2,p^r)$	$p^r\,(p^{2r}-1)$	$p^r\,(p^{2r}-1)$	1
$PGL(2,p^r)$	$p^r\,(p^{2r}-1)$	$p^r\,(p^{2r}-1)$	1
H	$e\,(e^2-1)$	$e\,(e^2-1)$	1

Table 3: Table of marks of $H = PGL(2, e)$ when $e \equiv 1 \ mod\ 4$

	I	$C_2(1)$	$C_2(2)$	$D_4(1)$	$D_4(2)$	C_{e-1}	P_e	C_{e+1}	$D_{2(e-1)}$	$D_{2(e+1)}$	H
$H(/I)$	$e(e^2-1)$										
$H(/C_2(1))$	$\frac{e\left(e^2-1\right)}{2}$	$e-1$									
$H(/C_2(2))$	$\frac{e\left(e^2-1\right)}{2}$	0	$e+1$								
$H(/D_4(1))$	$\frac{e\left(e^2-1\right)}{4}$	$\frac{3(e-1)}{2}$	0	6							
$H(/D_4(2))$	$\frac{e\left(e^2-1\right)}{4}$	$\frac{e-1}{2}$	$e+1$	0	2						
$H(/C_{e-1})$	$e(e+1)$	2	0	0	0	2					
$H(/P_e)$	e^2-1	0	0	0	0	0	$e-1$				
$H(/C_{e+1})$	$e(e-1)$	0	2	0	0	0	0	2			
$H(/D_{2(e-1)})$	$\frac{e(e+1)}{2}$	$\frac{e+1}{2}$	$\frac{e+1}{2}$	3	1	1	0	0	1		
$H(/D_{2(e+1)})$	$\frac{e(e-1)}{2}$	$\frac{e-1}{2}$	$\frac{e+3}{2}$	0	2	0	1	0	0	1	
$H(/H)$	1	1	1	1	1	1	1	1	1	1	1

Table 4: Table of marks of $H = PGL(2,e)$ when $e \equiv -1 \ mod \ 4$

	I	$C_2(1)$	$C_2(2)$	$D_4(1)$	$D_4(2)$	C_{e-1}	P_e	C_{e+1}	$D_{2(e-1)}$	$D_{2(e+1)}$	H
$H(/I)$	$e(e^2-1)$										
$H(/C_2(1))$	$\frac{e(e^2-1)}{2}$	$e-1$									
$H(/C_2(2))$	$\frac{e(e^2-1)}{2}$	0	$e+1$								
$H(/D_4(1))$	$\frac{e(e^2-1)}{4}$	$e-1$	$\frac{e+1}{2}$	2							
$H(/D_4(2))$	$\frac{e(e^2-1)}{4}$	0	$\frac{3(e+1)}{2}$	0	6						
$H(/C_{e-1})$	$e(e+1)$	2	0	0	0	2					
$H(/P_e)$	e^2-1	0	0	0	0	0	$e-1$				
$H(/C_{e+1})$	$e(e-1)$	0	2	0	0	0	0	2			
$H(/D_{2(e-1)})$	$\frac{e(e+1)}{2}$	$\frac{e+1}{2}$	$\frac{e+1}{2}$	2	0	1	0	0	1		
$H(/D_{2(e+1)})$	$\frac{e(e-1)}{2}$	$\frac{e-1}{2}$	$\frac{e+3}{2}$	1	3	0	1	0	0	1	
$H(/H)$	1	1	1	1	1	1	1	1	1	1	1

Let M be Table 3 or 4, $Q = (Q_1, Q_2, \ldots, Q_{11})$ and
$R = \left(\frac{(q^2-1)}{e(e^2-1)}, \frac{e(q-1)}{e^2-1}, \frac{e(q-1)}{e^2-1}, 4, 4, \frac{q-1}{e-1}, \frac{q+1}{e+1}, \frac{q}{e}, 2, 2, 1 \right)$.
By Theorem 2.6, $M^T Q^T = R^T$. It follows that,
$Q = (\frac{(e^2-q)(e^4+e^3-e^2q+e^2+e-q^2)}{e^2(e^2-1)^2}, \frac{q-e^2}{e^2-1}, \frac{q-e^2}{e^2-1}, 0, 0, \frac{q-2e+1}{2(e-1)}, \frac{q-2e-3}{2(e+1)}, \frac{q-e}{e(e-1)}, 1, 1, 1)$.
By Theorems 2.6 and 2.1, the subdegrees of this action are displayed in Table 1.
From Table 1, the rank is given by,

$$R(G) = \frac{e^5q - e^5 + e^4q - e^3q + e^3 - 4e^2q + 2e^2 + q^3}{e^2(e^2-1)^2}. \tag{7}$$

□

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